Άσκηση 2.7 (σελ. 60 βιβλίο)

$X_1, X_2, \ldots, X_n$ τδ από την $B(1,\theta)$

$p(x|\theta) = \theta^x \cdot (1-\theta)^{1-x}, \; x=0,1 \quad n=5$

$H_0: \theta = \frac{1}{2}$ vs $H_1: \theta = \frac{3}{4}$

α) Να βρεθεί ισχυρότατο τέστ επιπέδου σημαντικότητας α: $1\% \leq \alpha \leq 5\%$.

β) Ποιο το Minmax τέστ όταν $L(\theta_0, d_1) = 1$, $L(\theta_1, d_0) = 2$

γ) Bayes τέστ $\Pi(\theta_0) = \frac{1}{3}$, $\Pi(\theta_0) = \frac{2}{3}$.

ΛΥΣΗ:

$$\lambda = \frac{\prod_{i=1}^{n} f(x_i|\theta_0)}{\prod_{i=1}^{n} f(x_i|\theta_1)} = \frac{\theta_0^{\Sigma x_i} \cdot (1-\theta_0)^{n-\Sigma x_i}}{\theta_1^{\Sigma x_i} \cdot (1-\theta_1)^{n-\Sigma x_i}} \le k \Rightarrow$$

$$\Rightarrow \left(\frac{\theta_0}{(1-\theta_0)} \cdot \frac{(1-\theta_1)}{\theta_1} \right)^{\Sigma x_i} \le \frac{k}{(1-\theta_0)^n} (1-\theta_1)^n \Rightarrow$$

$$\Rightarrow \Sigma x_i \cdot \log\left(\frac{\theta_0}{(1-\theta_0)} \cdot \frac{(1-\theta_1)}{\theta_1} \right) \le \log\left(\frac{k}{(1-\theta_0)^n} \cdot (1-\theta_1)^n \right) \Rightarrow$$

$$\Rightarrow \Sigma x_i \ge \frac{\log\left(\frac{k}{(1-\theta_0)^n} \cdot (1-\theta_1)^n \right)}{\log\left(\frac{\theta_0}{(1-\theta_0)} \cdot \frac{(1-\theta_1)}{\theta_1} \right)} = \frac{\log\left(\frac{k}{(1-\theta_0)^n} \cdot (1-\theta_1)^n \right)}{\log\left(\frac{1}{3}\right)} = k^*$$

ii) $a = P(\text{Απορ. } H_0 \mid H_0 \text{ Αληθ.}) = P(\Sigma x_i \ge k^* / x_i \sim B(1,\theta_0)) =$
$= P(\Sigma x_i \ge k^* / x_i \sim B(1, 0.5)) = P(\Sigma x_i \ge k^* / \Sigma x_i \sim B(n, 0.5))$
$= P(\Sigma x_i \ge k^* / \Sigma x_i \sim B(5, 0.5)) =$
$= 1 - P(\Sigma x_i \le k^* - 1 / \Sigma x_i \sim B(5, 0.5)) =$
$= 1 - 0,9688 = 0,312$ (ισχύει)
Τότε το $k^* - 1 = 4 \Rightarrow k^* = 5$.

Η ισχύς είναι

$\gamma = 1 - P(\text{Αποδέχομαι την } H_0 \mid H_a \text{ Αληθής}) =$
$= P(\text{Απορ. } H_0 \mid H_a \text{ Αληθής}) =$
$= P(\Sigma x_i \ge 5 / \Sigma x_i \sim B(5, 0.75)) =$
$= P(\Sigma x_i = 5 / \Sigma x_i \sim B(5, 0.75)) = 0,75^5$

ή β' τρόπος
$0,75^5 \rightarrow 5$ επιτυχίες με $P(E) = 0.75$
(αν πχ $P(\Sigma x_i \ge 5 / \Sigma x_i \sim B(5, 0.75)) \longrightarrow 5$ αποτυχίες με $P(A) = 0,25$
ισοδύναμα 0 επιτυχίες με $P(E) = 0,75$

Επομένως, $P(Y=0) = \binom{5}{0} \cdot (1-p)^0 \cdot (1-(1-p))^5$ με $p = 0,75$.

ii) $\overset{1}{\cancel{L(\theta_0,d_1)}} \cdot P(\Sigma X_i \geq k^* \mid \Sigma X_i \sim B(5, 0.5)) = \overset{2}{\cancel{L(\theta_1,d_0)}} \cdot P(\Sigma X_i \leq k^*-1 \mid \Sigma X_i \sim B(5, 0.75))$

k^*	Αριστερό Μέρος	Δεξιό Μέρος
0	1	$2 \cdot 0 = 0$
1	$1-0{,}0313=0{,}9688$	$2 \cdot P(\Sigma X_i=0) = 2\binom{5}{0} \cdot 0{,}75^0 \cdot 0{,}25^5 = 2 \cdot 0{,}25^5 = 2(1-0{,}9990)$
2	$1-0{,}1875=0{,}8125$	$2P(\Sigma X_i \leq 1) = 2\left(P(\Sigma X_i=0)) + P(\Sigma X_i=1)\right) = 2 \cdot \left((1-0{,}9990)+(0{,}9990-0{,}9844)\right) = \ldots$
3	$1-0{,}5=0{,}5$	$\vdots$
4	$1-0{,}8125=0{,}1875$	$\vdots$
5	$1-0{,}9688=0{,}0313$	$\vdots$

Θα βγει το $k^*=3$

iii) Άσκηση
(Απάντηση: $\overset{\text{κρ.περ.}}{\Sigma X_i \geq 2}$ και το $k^* = \left(\frac{4}{2}\right)^5$)

Τεστ Πηλίκου Πιθανοφανειών (LRT) (Εκτός Ύλης)

Ας είναι $X_1, X_2, \ldots, X_n$ τδ από πληθυσμό με σππ ή σπ $f(x|\theta)$, όπου $\theta \in \Theta$. Θέλαμε να ελέγξουμε την

$H_0: \theta \in \Theta_0 \subset \Theta$ vs $H_a: \theta \in \Theta_a$

$$\lambda = \frac{\max_{\theta \in \Theta_0} L(\theta|x)}{\max_{\theta \in \Theta} L(\theta|x)} \leq k$$

Παρ 1

$X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ με μ, σ^2: άγνωστα

$H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$

$f(x|\theta) = f(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma^2 > 0$

Έστω το $\mu = \theta_1$ και $\sigma^2 = \theta_2$

Άρα, $f(x|\theta) = f(x|\theta_1,\theta_2) = (2\pi\theta_2)^{-\frac{1}{2}} e^{-\frac{1}{2\theta_2}(x-\theta_1)^2}$

$L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta) = (2\pi\theta_2)^{-n/2} e^{-\frac{1}{2\theta_2}\Sigma(x_i-\theta_1)^2}$

$\max_{\theta \in \Theta} L(\theta|x) = ;$ ή $\max_{\theta \in \Theta} \log L(\theta|x) = ;$

ώστε παραγωγίζοντας μερικώς $\Rightarrow \hat{\theta}_1 = \bar{x}$ και $\hat{\theta}_2 = \frac{1}{n}\Sigma(x_i-\bar{x})^2$

$$\max_{\theta \in \Theta_0} L(\theta|x) = ;$$

$$L(\theta_0|x) = \prod_{i=1}^{n} f(x_i|\theta) = (2\pi\theta_2)^{-n/2} \cdot e^{-\frac{1}{2\theta_2}\Sigma(x_i-\theta_0)^2}$$

$$\Rightarrow \tilde{\theta}_2 = \frac{1}{n}\Sigma(x_i-\theta_0)^2 .$$

Άρα,

$$\lambda = \frac{(2\pi\tilde{\theta}_2)^{-\frac{n}{2}} \cdot \cancelto{e^{-n/2}}{e^{-\frac{1}{2\tilde{\theta}_2}\Sigma(x_i-\theta_0)^2}}}{(2\pi\hat{\theta}_2)^{-\frac{n}{2}} \cdot \cancelto{e^{-n/2}}{e^{-\frac{1}{2\hat{\theta}_2}\Sigma(x_i-\bar{x})^2}}} = \left(\frac{\hat{\theta}_2}{\tilde{\theta}_2}\right)^{n/2} =$$

$$= \left(\frac{\frac{1}{n}\Sigma(x_i-\bar{x})^2}{\frac{1}{n}\Sigma(x_i-\theta_0)^2}\right)^{n/2} \leq k \Rightarrow$$

$$\Rightarrow \cancel{\left[\Sigma(x_i-\bar{x})^2 - \Sigma(x_i-\theta_0)^2\right]^{n/2} \leq k} \Rightarrow$$

$$\Rightarrow \cancel{\Sigma(x_i-\bar{x})^2 - \Sigma(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2}$$

$$\frac{\Sigma(x_i-\bar{x})^2}{\Sigma(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2} \leq k \Rightarrow$$

$$\Rightarrow \left(\frac{1}{1+\frac{n(\bar{x}-\theta_0)^2}{\Sigma(x_i-\bar{x})^2}}\right)^{n/2} \leq k \Rightarrow$$

$$\Rightarrow \frac{n(\bar{x}-\theta_0)^2}{\Sigma(x_i-\bar{x})^2} \geq k^*$$

$$\alpha = P\left(\frac{n(\bar{x}-\theta_0)^2}{\Sigma(x_i-\bar{x})^2} \geq k^* \,/\, X_i \sim N(\theta_0, \sigma^2)\right)$$

$$\bar{X} \sim N\left(\theta_0, \frac{\sigma^2}{n}\right) \Rightarrow \frac{(\bar{X}-\theta_0)}{\sigma/\sqrt{n}} \sim N(0,1) \Rightarrow \frac{(\bar{X}-\theta)^2}{\sigma^2/n} \sim \chi_1^2$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Coef

$$\frac{\dfrac{(\bar{X}-\mu_0)^2}{\sigma^2/n}}{\dfrac{(n-1)S^2}{\sigma^2}\Big/(n-1)} \sim F_{1,n-1}$$